COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 14 (2) (1994) 159–165

ON THE DIVISIBILITY OF HOMOGENEOUS HYPERGRAPHS

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Received June 4, 1991

We denote by K_ℓ^k, k, $\ell \geq 2$, the set of all k-uniform hypergraphs K which have the property that every ℓ element subset of the base of K is a subset of one of the hyperedges of K. So, the only element in K_2^2 are the complete graphs. If $\mathcal{T}$ is a subset of K_ℓ^k then there is exactly one homogeneous hypergraph $H_{\mathcal{T}}$ whose age is the set of all finite hypergraphs which do not embed any element of $\mathcal{T}$. We call $H_{\mathcal{T}}$ the τ-free homogeneous hypergraph. The K_n-free homogeneous graphs H_n have been shown to be indivisible, that is, for any partition of H_n into two classes, one of the classes embeds an isomorphic copy of H_n. [5]. Here we will investigate this question of indivisibility in the more general context of $\mathcal{T}$-free homogeneous hypergraphs. We will derive a general necessary condition for a homogeneous structure to be indivisible and prove that all $\mathcal{T}$-free hypergraphs for $\mathcal{T} \subset K_\ell^k$ with $\ell \geq 3$ are indivisible. The $\mathcal{T}$-free hypergraphs with $\mathcal{T} \subset K_2^k$ satisfy a weaker form of indivisibility which was first shown by Henson [2] to hold for H_n. The general necessary condition for homogeneous structures to be indivisible will then be used to show that not all $\mathcal{T}$-free homogeneous hypergraphs are indivisible.

1. Introduction

We will only deal with finite or countably infinite structures. The relational structure R is a *substructure* of the relational structure S is the set of elements of R is a subset of the set of elements of S and the identity function is and embedding from R into S. If S is a relational structure, then the $age(S)$, is the set of all finite substructures of S considered up to isomorphism. The relational structure S is *indivisible*, (negation *divisible*), if for every partition of S into two classes C and D, there is an embedding of S into C or into D. The relational structure S is *weakly indivisible* if for every partition of S into two classes C and D with $age(C) \neq age(S)$, there is an embedding of S into D. The *age of S is indivisible* if for every partition of S into finitely many classes the age of one of those classes is equal to the age of S. Clearly, if S is indivisible, then S is weakly indivisible and if S is weakly indivisible, then the age of S is indivisible. The notion of indivisible structure has been investigated by Fraïssé [1]. In [2] that the K_n-free homogeneous graphs are weakly indivisible. Pouzet [3] investigated structures with indivisible age and relates this property to certain Ramsey-type theorems. In [4] we further investigated this connection with the Ramsey theory. The relational structure S has the *embedding* property if for every $A \in age(S)$ and $x \in A$ and embedding φ: $A - x \to S$ there exists an extension φ' of φ which is an embedding from $A \to S$. The embedding property implies that if $T \in age(S)$ and $A \subset T$ and φ is an embedding

This research has been supported by NSERC grant 69–1325.

AMS subject classification code (1991): 04 A 20, 05 C 55

from A into S, then φ has an extension to an embedding of T into S. Relational structures which have the embedding property are called *homogeneous* [1, p. 313].

A set $\mathcal{R}$ of finite structures with a common language $\mathcal{L}$ is an *age* if $\mathcal{R}$ is closed under substructures, isomorphic images and the property that for any two elements A, $B \in \mathcal{R}$ there exists an element $C \in \mathcal{R}$ together with embeddings from A into C and B into C. It is known, [1,273], that a set $\mathcal{R}$ of finite relational structures is an ege if and only if there is a countable $\mathcal{L}$-structure S such that $age(S) = \mathcal{R}$. The set of relational structures $\mathcal{R}$ has the *amalgamation property* if for any three elements $A, B, C \in \mathcal{R}$ and embeddings $f : C \to A$ and $g : C \to B$, there is a structure $D \in \mathcal{R}$ and embeddings $f_1 : A \to D$ and $g_1 : B \to D$ such that $f_1 \circ f = g_1 \circ g$ holds. In [1, 313–317] the following three statements are proven: If S is a homogeneous structure the $age(S)$ has the amalgamation property. If the set $\mathcal{R}$ of relational structures is an age and has the amalgamation property then there is a unique homogeneous structure S with $age(S) = \mathcal{R}$. If S is a homogeneous structure, A and B are two finite substructures of S and α is an isomorphism from A to B then there is an automorphism β of S which extends α and conversely, every relational structure with the above property is homogeneous. It is often convenient to describe the age of homogeneous structure S by a set of forbidden substructures. The homogeneous graphs are indivisible. Using the classification in [6] it becomes apparent that the difficulty lies in showing whether the homogeneous k_n-free graphs H_n are indivisible. In [7] Komjáth and Rödl proved that H_3 is indivisible. We extended their result by showing that for every $n \in \omega$, H_n is indivisible [5].

Let for $K, \ell \geq 2$, K_ℓ^k denote the set of all finite k-uniform hypergraphs K which have the property that every ℓ element subset of the base of K is a subset of one of the hyperedges of K. Let $\mathcal{I}$ be a subset of K_ℓ^k. We call such a set $\mathcal{I}$ an ℓ covering set of hypergraphs with arity k. Denote by $Forb(\mathcal{I})$ the set of all k-uniform hypergraphs G such that $age(G) \cap \mathcal{I} = \emptyset$. It is easy to see that $Forb(\mathcal{I})$ is an age and that $Forb(\mathcal{I})$ has the amalgamation property. Hence there exists then exactly one homogeneous hypergraph $H_\mathcal{I}$, the $\mathcal{I}$-free homogeneous hypergraph, such that for all finite k-uniform hypergraphs A, $A \in age(H_\mathcal{I})$ holds if and only if $age(A) \cap \mathcal{I} = \emptyset$.

In this paper we are primarily concerned with divisibility properties of the homogeneous hypergraphs of the form $H_\mathcal{I}$. It turns out that Theorem 1 has a much wider scope then just the homogeneous hypergraphs $H_\mathcal{I}$. We will therefore formulate it for homogeneous structures over some relational language $\mathcal{L}$. If R and S are two relational structures over the same language $\mathcal{L}$, we say that R *precedes* S, or $R \preceq S$, if and only if there are finitely many $\mathcal{L}$-structures R_i, $i \in n$, with embeddings $\alpha_i : R_i \to R$ and $\beta_i : R_i \to S$ such that for each element $x \in R$ there is an $i \in n$ with $x \in \alpha_i(R_i)$. Note that if $R \preceq S$ holds then R can be partitioned into finitely many pieces each of which can be embedded into S and that the relation $\preceq$ is transitive. If $\mathcal{R}$ is a set of $\mathcal{L}$-structures, we say that $\mathcal{R}$ satisfies the *chain condition* if for every two elements R and S at least one of $R \preceq S$ or $S \preceq R$ holds.

If S is a relational structure and $T \subset S$, we say that T is a *finitely induced orbit* of S if there is some finite $A \subset S$ such that T is an orbit of the group of all automorphisms of S which stabilize A element by element. Note that if S is a homogeneous structure then T is, in the language of model theory, a set of *one types over* A. That is, every element of T is via the relations of S "attached" to A in the same way as any other element of T. If T is a finitely induced orbit of

S we will, by misusing the notation, use T as well to denote the substructure of S induced by T. We will prove:

Theorem 1. *If the homogeneous structure S is indivisible, then the finitely induced orbits of S satisfy the chain condition.*

Theorem 2. *If $\ell \geq 3$ and $k \geq 3$ then for every ℓ covering set $\mathcal{I}$ of hypergraphs with arity k, the $\mathcal{I}$-free homogeneous hypergraph $H_{\mathcal{I}}$ is indivisible .*

Theorem 3. *If $\ell \geq 2$ and $k \geq 2$ then for every ℓ covering set $\mathcal{I}$ of hypergraphs with arity k, the $\mathcal{I}$-free homogeneous hypergraph $H_{\mathcal{I}}$ is weakly indivisible.*

Theorem 4. *For $\ell, k \geq 2$ an ℓ-covering set $\mathcal{I}$ of hypergraphs with arity k, every finitely induced orbit of $H_{\mathcal{I}}$ is weakly indivisible.*

The homogeneous structure $H_{\mathcal{I}}$ itself is an orbit of the empty set and hence Theorem 3 is a special case of Theorem 4. Theorem 4 follows from Theorem 7 in [4]. This leaves us to decide for which 2 covering sets $\mathcal{I}$ of hypergraphs the homogeneous hypergraph $H_{\mathcal{I}}$ is indivisible. As this proble, is completely solved for ordinary graphs, we will only be interested in hypergraphs of arity larger than or equal to 3. Observe here that Theorem 1 and Theorem 4 together imply the following statement:

Theorem 5. *If the homogeneous hypergraph $H_{\mathcal{I}}$, for some ℓ covering set $\mathcal{I}$ of hypergraphs with arity k, is indivisible, then the ages of the finitely induced orbits of $H_{\mathcal{I}}$ are totally ordered under set inclusion.*

Let $\mathcal{L}$ be a relational language and S a homogeneous model of $\mathcal{L}$. (That is a relational structure with relational symbols from $\mathcal{L}$). We say a pair (A,x) is a type of S if $A \in age(S)$ and $x \in A$. If $B \in age(S)$ and T is a model of $\mathcal{L}$, then T is an $[(A,x),B]$-structure is there exist embeddings $\varphi : A - x \to T$ and $\psi : B \to T - \varphi(A-x)$ with $\varphi(A-x) \cup (\psi(B) = T$ such that for all $b \in \psi(B)$ there is an extension φ' of φ with $\varphi'(x) = b\}$. The (A,x) derived set of S, $D_S(A,x) = D(A,x)$ is: $D)A,x) = \{B \in age(S)$, such that no $[(A,x),B]$-structure is an element of $age(S)\}$.

Theorem 6. *The ages of the finitely induced orbits of the homogeneous structure S are totally ordered under setinclusion if and only if the set of derived sets of S is totally ordered under setinclusion.*

In those cases where the finitely induced orbits of S are age indivisible, Theorem 6 together with Theorem 1 constitutes a mild improvement over Theorem 1 alone. We can show that the derived sets of S are not totally ordered by exhibiting a finite number of finite models of $\mathcal{L}$. If S is given by a finite number of excluded substructures, we do not know in general whether there is an effective procedure to check if the derived sets are totally ordered. Even in the case of homogeneous structures of the form $H_{\mathcal{I}}$ this problem is still open. If $\mathcal{L}$ consists only of binary relational symbols there is an effective procedure.

We will construct a finite set $\mathcal{I}$ of two covering graphs of arity 3 and exhibit two derived sets for $H_{\mathcal{I}}$ which are not comparable under set inclusion. This then will prove in contrast to the case of ordinary graphs:

Theorem 7. *Not all homogeneous hypergraphs of the form $H_{\mathcal{I}}$ are indivisible.*

2. Proof of Theorem 1

Assume that $A, B \subset S$ are two finite subsets such that A induces the orbit T and B induces the orbit R. Our aim, is to prove that if T and R are not comparable by $\preceq$ then the homogeneous structure S is divisible. If T is finite we can split T into finitely many singletons and embed each of those singletons onto R, unless S has a unary relation which holds for some elements of T but for none of the elements of R. But clearly in this case the homogeneous structure S is divisible. Hence we will assume that both finitely induced orbits T and R are infinite. We will also assume that the base set of S is the set of natural numbers. If φ is a local automorphism from A, (resp. B), into S and ψ an extension of φ to an automorphism of S, then T_φ, (R_φ), is the set of all $x \in \psi(T)$, $(x \in \psi(R))$, which are larger than all the elements in A, or B respectively. Note that T_φ, (R_φ), is independent of the particular extension ψ of φ. The finite subsets of S are totally ordered by putting $X < Y$ iff the largest element in the symmetric difference of X and Y is an element of Y. In order to prove that S is divisible, we have to find a partition of S into finitely many classes, none of which containing an isomorphic copy of S. We claim that the following partition of S has this property:

C_1 is the set of all $x \in S$ that there is no embedding $\varphi : A \to S$ with $x \in T_\varphi$.

C_2 is the set of all $x \in S - C_1$ such that there is no embedding $\varphi : B \to S$ with $x \in R_\varphi$.

C_3 is the set of all $x \in S - C_1 - C_2$ such that there is an embedding $\varphi : A \to S$ and $x \in T_\varphi$ with the property that for every embedding $\psi : B \to S$ with $x \in R_\psi$, $\varphi(A) < \psi(B)$.

C_4 is the set of all $x \in S - C_1 - C_2$ such that there is an embedding $\psi : B \to S$ and $x \in R_\psi$ with the property that for every embedding $\varphi : A \to S$ with $x \in T_\varphi$, $\psi(B) < \varphi(A)$.

If there were an embedding $\psi : S \to C_1$ then the restriction pf Ψ to A is an embedding from A into S. Because T is infinite, there is some $x \in \psi(T)$ with $\{x\} > \psi(A)$. Clearly, $x \in T_\psi$ in contradiction to the definition of C_1.

For the same reason C_2 does not contain an isomorphic copy of S.

Next, assume that $\varphi : S \to C_3$ is an embedding. The restriction of φ to B is an embedding of B into S. Observe that $R_\varphi \cap C_3$ consists of those elements of $\varphi(R)$ which are above $\varphi(B)$. Because of the definition of C_3, we must have the following: To each $x \in R_\varphi \cap C_3$ there exists an automorphism ψ of S such that $\psi(A) < \varphi(B)$ and $x \in \psi(T)$. There are only finitely many sets of the form $\psi(A)$ with $\psi(A) < \varphi(B)$. This partitions R_φ into finitely many pieces each of them a subset of T_ψ and hence of $\psi(T)$ for some automorphism ψ with $\psi(A) < \varphi(B)$. There are only finitely many elements in $R - R_\varphi$. But this now yields a partition of $\varphi(R)$ and hence of R into finitely many pieces each of which can be embedded into T.

Clearly, by just trading the roles of A and B, this same argument can be used to show that C_4 does not contain an isomorphic copy of S. ∎

3. Proof of Theorem 2

For $\ell \geq 3$, let $\mathcal{J}$ be an ℓ-covering set of graphs and $H_{\mathcal{J}}$ the corresponding $\mathcal{J}$-free homogeneous graph. We assume that the base set of $H_{\mathcal{J}}$ is the set of natural numbers. An initial segment of $H_{\mathcal{J}}$ is the subgraph of $H_{\mathcal{J}}$ induced by some initial segment of the natural numbers. We note first, That if $F \subset H_{\mathcal{J}}$ is finite, then $H_{\mathcal{J}} - F$ is isomorphic to F. First of all, $age(H_{\mathcal{J}}) = age(H_{\mathcal{J}} - f)$ because for all $\mathcal{J}$-free hypergraphs A, the disjoint union of F and A is again $\mathcal{J}$-free. Using the embedding property we see that $F \cup A$ can be embedded into $H_{\mathcal{J}}$ by an embedding φ which is the identity on F.

Secondly, if $A \in age(H_{\mathcal{J}}$ and $x \in A$ and φ an embedding from $A - x$ into $H_{\mathcal{J}} - F$, let B be the subgraph of $H_{\mathcal{J}}$ induced by $F \cup \varphi(A-x)$. Let i be the identity function $A - x$. There exists then an amalgam D of B and A and functions $f: A \to D$ and $g: B \to D$ such that $g \circ \varphi = f \circ id$ and $f(x) \notin g(F)$ [1]. The function g^{-1} is a partial embedding from D into $H_{\mathcal{J}}$ which, by the embedding property, can be extended to an embedding ψ of D into $H_{\mathcal{J}}$. $\psi(f(x)) \in H_{\mathcal{J}} - F$ is then an extension of φ to an embedding of A into $H_{\mathcal{J}} - F$. Hence the structure $H_{\mathcal{J}} - F$ has the same age as $H_{\mathcal{J}}$ and is homogeneous. This implies then that the two homogeneous structures $H_{\mathcal{J}} - F$ and $H_{\mathcal{J}}$ are isomorphic.

Let now B (blue) and R (red) be a partition of $H_{\mathcal{J}}$ into two classes. Let φ be an embedding of some initial segment I_n into R which can not be extended to an embedding of I_{n+1} into R. If such a φ does not exist, we can construct an embedding of $H_{\mathcal{J}}$ into R. Similarly, there is an embedding ψ of an initial segment I_m into B which can not be extended to an embedding of I_{m+1}. Let D be the subset of $H_{\mathcal{J}}$ induced by $\varphi(I-n) \cup \psi(I_m)$. We extend now D by a new point x and a minimal set of hyperedges, such that $\varphi^{-1} : \varphi(I_n) \to I_n$ can be extended to an embedding f with $f(x) = n+1$ and $\psi^{-1} : \psi(I_m) \to I_m$ can be extended to an embedding g with $g(x) = m+1$. Observe that $D \cup \{x\}$ does not contain an element of $\mathcal{J}$ and hence $D \cup \{x\} \in age(H_{\mathcal{J}})$. Hence, the identity map from $D \to D$ can be extended to an embedding $\alpha : D \cup \{x\} \to H_{\mathcal{J}}$. But this is a contradiction to the stipulation of φ and ψ being maximal. ∎

4. Proof of Theorem 6

Let $\mathcal{L}$ be a relational language and S a homogeneous model of $\mathcal{L}$. If (A,x) is a type of S and $\varphi : A - x \to S$ an embedding consider the set $R = \{\psi(x); \psi$ is an embedding from $A \to S$ which is an extension of $\varphi\}$. Obviously R is a finitely induced orbit of S and is called the orbit induced by the type (A,x). (R depends on φ but only up to isomorphism). On the other hand, if R is finitely induced by the finite subset $F \subset S$, and $x \in R$ any element, and A the substructure of S induced by $F \cup \{x\}$, then R is the induced orbit of the type (A,x). We claim now that $B \in D(A,x)$ iff $B \notin age(R)$ when R is the orbit induced by (A,x). We assume without loss that $A \subset S$ and $B \subset R$. Then the substructure T of S induced by $A \cup B$ is clearly an $[(A,x),B]$-structure and $T \in age(S)$, so $B \notin D(A,x)$. If $B \notin D(A,x)$ there is some $[(A,x),B]$-structure $T \in age(S)$. If φ is an embedding from $A - x$ into S then φ has an extension φ' to T. Clearly then $\varphi'(B)$ must be a subset of the orbit induced by (A,x) under φ. This means that the derived sets are just the complimentary sets of the ages of the finitely induced orbits, a fact which obviously establishes Theorem 5. ∎

5. An example of a divisible homogeneous structure of the form $H_{\mathcal{J}}$

All the hypergraph and hyperedges in this chapter will have arity 3. Let F be the hypergraph with $V(F)=\{X_1,x_2,x_3,x_4,x_5\}$ and $E(F)=\{\{X_1,x_2,x_3\}, \{x_4,x_5,x_1\}, \{x_4,x_5,x_2\}, \{x_4,x_5,x_3\}\}$. $V(K)=\{y_1,y_2,y_3,y_4\}$ and $E(K)$ is the set of all 3 element subsets of $V(K)$. Observe first that there is no embedding from K into F or F into K and that both F and K are 2-covering.

$\mathcal{J}_1$ is the set of hypergraphs T such that $V(T)=V(F)\cup V(K)$ and $\mathcal{A}_1\subset E(T)\subset \mathcal{B}_1$ where:

$\mathcal{A}_1=E(F)\cup E(K)\cup\{S;|S|=3\wedge S\subset V(T)\wedge|S\cap V(F)|=1\}$

$\mathcal{B}_1=E(F)\cup E(K)\cup\{S;|S|=3\wedge S\subset V(T)\wedge 1\leq|S\cap V(F)|\leq 2\}$.

$\mathcal{J}_2$ is the set of hypergraphs T such that $V(T)=V(F)\cup C(K)$ and $\mathcal{A}_2\subset E(T)\subset \mathcal{B}_2$, where:

$\mathcal{A}_2=E(F)\cup E(K)\cup\{S;|S|=3\wedge S\subset V(T)\wedge|S\cap V(K)|=1\}$.

$\mathcal{B}_2=E(F)\cup E(K)\cup\{S;|S|=3\wedge S\subset V(T)\wedge 1\leq|S\cap V(F)|\leq 2\}$.

We put $\mathcal{J}=\mathcal{J}_1\cup\mathcal{J}_2$ and consider $H_{\mathcal{J}}$.

The hypergraph A is constructed from F by adding one more vertex x and all three element subsets as edges, which contain x. The hypergraph B is constructed from K by adding one more vertex y and all three element subsets as edges, which contain y.

Observe that $D(A,x)$ is the set of all finite hypergraphs which contain K as an induced subgraph. Further, $D(B,y)$ is the set of all finite hypergraphs which contain F as an induced subgraph but $D(A,x)$ and $D(B,y)$ are not comparable under set inclusion. ∎

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